This problem was invented by Lewis Carroll in December, 1893 (S. D. Collingwood, *The Life and Letters of Lewis Carroll* (*Rev. C. L. Dodgson*), New York, 1899, pp. 317–318) and in his diary he remarked: "Got Professor Clifton's answer [R. B. Clifton, professor of physics at Oxford] to the 'Monkey and Weight Problem.' It is very curious, the different views taken by good mathematicians. Price [Bartholomew Price, professor of physics at Oxford] says that the weight goes *up* with increasing velocity; Clifton (and Harcourt [A. G. Vernon-Harcourt, professor of chemistry at Oxford]) that it goes *up*, at the same rate as the monkey; while Sampson [probably E. F. Sampson, lecturer, tutor and censor of Christ Church, Oxford] says that it goes *down*." Yet another solution by Rev. A. Brook is given on page 268 of *The Lewis Carroll Picture Book . . .* edited by S. D. Collingwood (London, 1899), namely that "the weight remains stationary."

The problem has been recently discussed in *School Science and Mathematics*, volume 17, December, 1917, p. 821; volume 19, December, 1919, p. 815; and volume 20, February, 1920, pp. 172–173. The editors of the MONTHLY invite mathematical solutions of the problem.

2839. By translating the steps of the construction of a regular pentagon from plane geometry into algebra show that one of the fifth roots of unity is equal to

$$\tfrac{1}{4}(\sqrt{5}-1)+\frac{i}{4}\sqrt{10+2\sqrt{5}}.$$

(This problem is proposed for solution in Wilczynski and Slaught, *College Algebra with Applications*, Boston, 1916, p. 193.)

2840. Proposed by NORMAN ANNING, University of Maine.

It is observed in a table of values of

$$\log_{10}(\operatorname{colog}_{10} x)$$

that second differences are zero for values of x in the neighborhood of 0.37. Prove that this must be the case. (Cf. Chappell's *Five-Figure Mathematical Tables*, Edinburgh, 1915, p. 180.)

2841. Proposed by WILLIAM HOOVER, Columbus, Ohio.

The mixed number,

$$9\frac{49}{64}=9+\frac{49}{64}=3^2+\frac{7^2}{8^2},$$

is of the type form

$$k^2+\frac{(2k+1)^2}{(2k+2)^2};$$

how may the *forms* of the terms of the fractional part be determined *deductively?*

Generally required that

$$k^2+\frac{\{\varphi_1(k)\}^2}{\{\varphi_2(k)\}^2}$$

be a perfect square; show how $\varphi_1(k)$ and $\varphi_2(k)$ may be found.

2842. Proposed by H. S. UHLER, Yale University.

Express explicitly the following sextic in x as the product of a quadratic and a biquadratic:

$$3x^6-6k_1x^5+(7k_1^2-9k_2^2)x^4-2(2k_1^3-4k_1k_2^2-3k_3^3)x^3+[(k_1^2-k_2^2)^2-9k_1k_3^3]x^2$$
$$-(k_1^2-2k_2^2)(k_1k_2^2-9k_3^3)x+(k_1^2-3k_2^2)(k_2^4-3k_1k_3^3).$$

SOLUTIONS OF PROBLEMS.

2746 [1919, 72]. Proposed by S. A. COREY, Des Moines, Iowa.

Establish the following algebraic identity without actually performing the indicated operations:

$$\begin{aligned}2(t_1t_2+c_1t_3t_4+c_2t_5t_6+c_1c_2t_7t_8)(r_1r_2+c_1r_3r_4+c_2r_5r_6+c_1c_2r_7r_8)\\
=(r_1t_1-c_1r_3t_3-c_2r_5t_5+c_1c_2r_7t_7)(r_2t_2-c_1r_4t_4-c_2r_6t_6+c_1c_2r_8t_8)\\
+(r_1t_2-c_1r_3t_4-c_2r_5t_6+c_1c_2r_7t_8)(r_2t_1-c_1r_4t_3-c_2r_6t_5+c_1c_2r_8t_7)\\
+c_1(r_1t_3+r_3t_1-c_2r_5t_7-c_2r_7t_5)(r_2t_4+r_4t_2-c_2r_6t_8-c_2r_8t_6)\end{aligned}$$

$$+ c_1(r_1t_4 + r_3t_2 - c_2r_5t_8 - c_2r_7t_6)(r_2t_3 + r_4t_1 - c_2r_6t_7 - c_2r_8t_5)$$
$$+ c_2(r_1t_5 + c_1r_3t_7 + r_5t_1 + c_1r_7t_3)(r_2t_6 + c_1r_4t_8 + r_6t_2 + c_1r_8t_4)$$
$$+ c_2(r_1t_6 + c_1r_3t_8 + r_5t_2 + c_1r_7t_4)(r_2t_5 + c_1r_4t_7 + r_6t_1 + c_1r_8t_3)$$
$$+ c_1c_2(r_1t_7 - r_3t_5 + r_5t_3 - r_7t_1)(r_2t_8 - r_4t_6 + r_6t_4 - r_8t_2)$$
$$+ c_1c_2(r_1t_8 - r_3t_6 + r_5t_4 - r_7t_2)(r_2t_7 - r_4t_5 + r_6t_3 - r_8t_1).$$

By assuming special relations between the constants involved show that the product of the sum of four squares by the sum of four squares equals the sum of four squares.

Solution by the Proposer.

We have the well-known identity:

$$(T_1^2 + c_1T_2^2 + c_2T_3^2 + c_1c_2T_4^2)(R_1^2 + c_1R_2^2 + c_2R_3^2 + c_1c_2R_4^2) = P_1^2 + c_1P_2^2 + c_2P_3^2 + c_1c_2P_4^2, \quad (1)$$

where

$$P_1 = T_1R_1 - c_1T_2R_2 - c_2T_3R_3 + c_1c_2T_4R_4, \qquad P_2 = T_2R_1 + T_1R_2 - c_2T_3R_4 - c_2T_4R_3,$$
$$P_3 = T_3R_1 + T_1R_3 + c_1T_4R_2 + c_1T_2R_4, \quad \text{and} \quad P_4 = T_4R_1 - T_1R_4 - T_3R_2 + T_2R_3.$$

Assuming R_n to be a vector (or even a quaternion), (1) will hold as to its scalar part, being a homogeneous quadratic.

Let $\theta = \sqrt{-1}$, $H = \frac{1}{2}[1 - j - \theta(i + k)]$, $I = \frac{1}{2}[1 + j + \theta(i - k)]$, $J = \frac{1}{2}[1 + j - \theta(i - k)]$, and $K = \frac{1}{2}[1 - j + \theta(i + k)]$, i, j, and k being Hamilton's unit vectors. Also let

$$R_1 = r_1H + s_1I + s_2J + r_2K, \qquad R_2 = r_3H + s_3I + s_4J + r_4K,$$
$$R_3 = r_5H + s_5I + s_6J + r_6K, \quad \text{and} \quad R_4 = r_7H + s_7I + s_8J + r_8K.$$

If $r_{2n-1} + s_{2n-1} + s_{2n} + r_{2n} = 0$, R_n becomes a vector but its tensor is usually imaginary, its square being $2(r_{2n-1}r_{2n} + s_{2n-1}s_{2n})$ (v. Vol. XIV, pp. 19–22 of MONTHLY). If $s_{2n} = 0$, this reduces to $2(r_{2n-1}r_{2n})$, and if we further assume that $s_{2n-1} = -(r_{2n-1}r_{2n})$, R_n becomes a vector the squared tensor (or norm) of which is $2(r_{2n-1}r_{2n})$. Substituting these vector values of R_1, R_2, R_3, and R_4 in P_1, P_2, P_3, and P_4, the norm of P_n is found to be $2p_{2n-1}p_{2n}$, where

$$p_1 = T_1r_1 - c_1T_2r_3 - c_2T_3r_5 + c_1c_2T_4r_7, \qquad p_2 = T_1r_2 - c_1T_2r_4 - c_2T_3r_6 + c_1c_2T_4r_8,$$
$$p_3 = T_2r_1 + T_1r_3 - c_2T_4r_5 - c_2T_3r_7, \qquad p_4 = T_2r_2 + T_1r_4 - c_2T_4r_6 - c_2T_3r_8,$$
$$p_5 = T_3r_1 + c_1T_4r_3 + T_1r_5 + c_1T_2r_7, \qquad p_6 = T_3r_2 + c_1T_4r_4 + T_1r_6 + c_1T_2r_8,$$
$$p_7 = T_4r_1 - T_3r_3 + T_2r_5 - T_1r_7, \quad \text{and} \quad p_8 = T_4r_2 - T_3r_4 + T_2r_6 - T_1r_8.$$

Substituting in (1) and reducing, we get

$$(T_1^2 + c_1T_2^2 + c_2T_3^2 + c_1c_2T_4^2)(r_1r_2 + c_1r_3r_4 + c_2r_5r_6 + c_1c_2r_7r_8)$$
$$= p_1p_2 + c_1p_3p_4 + c_2p_5p_6 + c_1c_2p_7p_8. \quad (2)$$

Similarly let $T_n = t_{2n-1}H + u_{2n-1}I + u_{2n}J + t_{2n}K$, where $u_{2n-1} = 0$ and $u_{2n} = -(t_{2n-1} + t_{2n})$. T_n will then be a vector.

Substituting these values of T_n in (2) and taking scalars the given identity is obtained. This process of taking scalars is simplified if we observe that if $Q_1 = x_1H + y_1I + z_1J + v_1K$, and $Q_2 = x_2H + y_2I + z_2J + v_2K$ are vectors, and if $y_1 = y_2 = 0$, $S \cdot Q_1Q_2 = -(x_1v_2 + v_1x_2)$.

If $c_1 = c_2 = 1$, $t_{2n-1} = t_{2n}$, and $r_{2n-1} = r_{2n}$ in the given identity, the product of the sum of four squares by the sum of four squares is seen to be the sum of four squares.

Further if t_{2n-1} be the conjugate of t_{2n}, and r_{2n-1} be the conjugate of r_{2n}, and if $c_1 = c_2 = 1$, in the given identity, it becomes 2 (sum of eight squares) (sum of eight squares) = (sum of 16 squares). This is a special case of

$$\left(\sum_{n=1}^{16} x_n^2\right)\left(\sum_{n=1}^{16} y_n^2\right) = \sum_{n=1}^{16} z_n^2,$$

and establishes the theorem:

The product of the sum of sixteen squares of numbers forming the legs of eight Pythagorean triangles by the sum of sixteen squares, eight of which have the same sum as the remaining eight, equals the sum of sixteen squares.

Proof of the given identity could have been obtained by the use of the ordinary complex quantities of algebra, but the above proof is given because of its greater generality and novelty.

It will be observed that the given identity as well as (1), (2), and the above 16-square theorem, all being homogeneous algebraic quadratic identities, may be given geometric interpretations by letting certain of the letters represent vectors and then taking the scalars of the resulting expressions.

2748 [1919, 72]. Proposed by J. B. REYNOLDS, Lehigh University.

The vertices of a triangle are (0, 0), (2a, 0), and (2x, 2y). Where are the vertices of the triangle of least area having its vertices on the perpendicular bisectors of the sides of the given triangle and the same center of gravity as the given triangle?

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $Q_1(x_1, y_1)$, $Q_2(x_2, y_2)$, $Q_3(x_3, y_3)$ be the vertices of the required triangle. Since Q_1, Q_2, Q_3, are on the perpendicular bisectors of the sides of the triangle $P_1P_2P_3$, we have

$$y_1y + x_1(x - a) - x^2 + a^2 - y^2 = 0, \tag{1}$$

$$y_2y + x_2x - x^2 - y^2 = 0, \tag{2}$$

$$x_3 = a. \tag{3}$$

If the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ have the same center of gravity

$$\frac{x_1 + x_2 + x_3}{3} = \frac{2x + 2a}{3}, \qquad \frac{y_1 + y_2 + y_3}{3} = \frac{2y}{3},$$

or

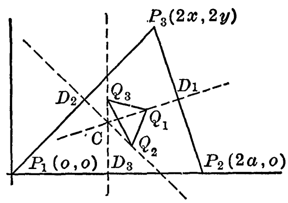
$$x_1 + x_2 = 2x + a, \tag{4}$$

$$y_1 + y_2 + y_3 = 2y. \tag{5}$$

From equations (1), (2), (4), (5), we find

$$\left.\begin{aligned} ax_1 &= -yy_3 + ax + a^2, \\ ax_2 &= yy_3 + ax, \\ ay_1 &= (x - a)y_3 + ay, \\ ay_2 &= -xy_3 + ay. \end{aligned}\right\} \tag{6}$$

The area of triangle $Q_1Q_2Q_3$ is given by



$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2a^2}\begin{vmatrix} ax_1 & ay_1 & 1 \\ ax_2 & ay_2 & 1 \\ ax_3 & ay_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2a^2}\begin{vmatrix} -yy_3 + ax + a^2, & (x - a)y_3 + ay, & 1 \\ yy_3 + ax, & -xy_3 + ay, & 1 \\ a^2, & ay_3, & 1 \end{vmatrix}$$

whence $2a\Delta = 3yy_3^2 - 2(x^2 + y^2 - ax + a^2)y_3 + a^2y$.

The area will be a minimum when $(d/dy_3)(2a\Delta) = 0$; that is, when

$$3yy_3 = x^2 + y^2 - ax + a^2.$$

It may be easily shown that the center of the circumcircle of $\Delta P_1P_2P_3$ is $C(a, y_0)$, where $yy_0 = x^2 + y^2 - ax$. Hence $3yy_3 = yy_0 + a^2$, or $y_3 = y_0/3 + a^2/3y$. The coördinates of the other vertices may now be found from equations (6).

Note. It may be shown that if the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ have the same center of gravity,

$$\frac{Q_1D_1}{P_2P_3} = \frac{Q_2D_2}{P_3P_1} = \frac{Q_3D_3}{P_1P_2},$$

where D_1, D_2, D_3 are the mid-points of the sides of Δ $P_1P_2P_3$. This property of the triangles might have been used in this problem.